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Theoretical investigation of the interfacial stability of inviscid fluids in motion, considering surface tension

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The present work is an analytical study of the stability of interfaces between fluids in motion, special attention being given to the role of surface tension without consideration of viscous effects. A variational approach based upon the principle of minimum free energy, which was first formulated for stagnant fluids, is applied to fluids in motion. This generalization is possible if viscous and inertia effects are unimportant as far as stability is concerned. One stability problem is studied in detail: a gas jet impinging on a free liquid. The analytical results obtained by this variational technique lie within the range of accuracy (15 %) of the experimental results for this gas-jet problem. The method is very general and therefore can be applied to quite a number of interface stability problems.

1. Introduction

The interface between two immiscible fluids in motion can become unstable under certain flow conditions. In recent years several authors have been concerned with this type of instability in connexion with a number of fluid flow problems. The present work reports an analytical study of these stability phenomena which was motivated by its possible application to the splattering of molten metal as observed during electric arc welding. The stability of a weld pool during high-current welding can best be studied by analysing the stability of the interface between a quiescent liquid and a gas jet impinging upon it (see Berghmans 1970). A gas jet impinging on a liquid has also been used as a simple model for the study of gas absorption into liquid systems by Mathieu (1960, 1962), because of its importance in metallurgy. Rosler & Stewart (1968) considered the impinging jet a good model for the study of liquid dispersion in more complex situations, while Turkdogan (1966) mentioned that it can be used to measure surface tension and to study the spreading of turbulent jets.

An axisymmetric gas jet impinging at right angles on a liquid at rest will cause an indention (see figure 1). Collins & Lubanska (1954), Banks & Chandrasekhara (1962), Turkdogan (1966) and Cheslak *et al.* (1969) studied experimentally the dependence of the maximum depth of penetration and the shape of the indention upon the type of liquid–gas combination, the jet strength, the jet diameter d_j and the separation H_j between the orifice and the undisturbed liquid. Mathieu (1960, 1962) in addition considered jets which did not impinge vertically upon the liquid surface. All these authors mentioned that the interface fluctuated at

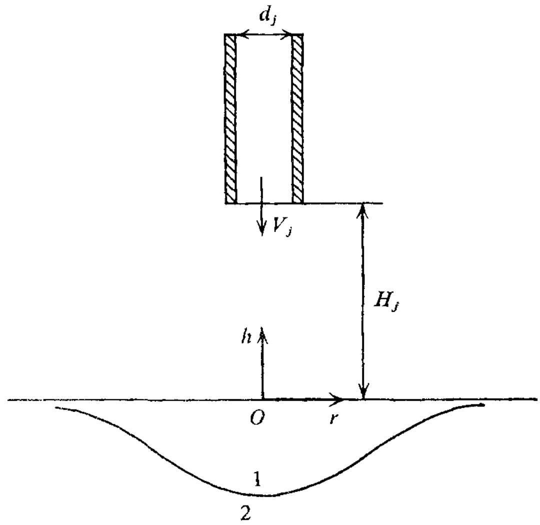

FIGURE 1. Gas jet impinging on a free liquid surface.

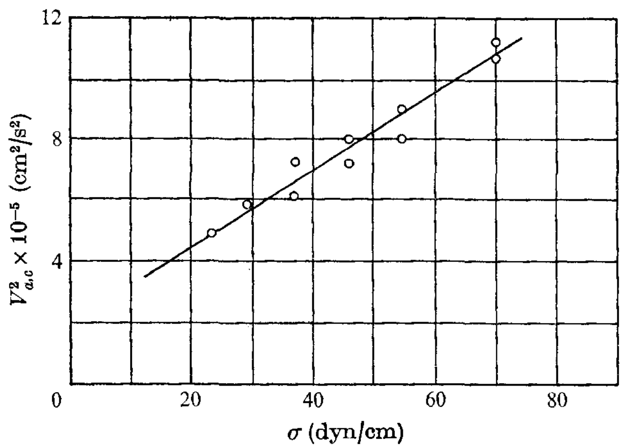

FIGURE 2. Critical jet velocity $V_{a,c}$ versus surface tension σ.
○, data of Rosler & Stewart (1968).

large jet strengths. In 1968 Rosler & Stewart published their work on the stability of the gas–liquid interface and observed that a critical average jet velocity $V_{a,c}$ exists at which the interface becomes unstable. Keeping the jet diameter and jet separation constant, they measured the critical average jet velocity as a function of surface tension (see figure 2). The oscillations of the interface become more vigorous when the jet velocity is further increased until a second type of instability develops which is characterized by dispersion of liquid droplets from the interface. The present work attempts to give a theoretical explanation of the first type of instability.

Banks & Chandrasekhara (1962) showed that, under the conditions in which instabilities were observed, the interfacial shear stress was too small to give rise to such phenomena. Rosler & Stewart (1968) pointed out that the motion of the liquid can be neglected, while their data clearly shows surface tension to be the important parameter involved.

The effects of surface tension upon the stability of the interface of two fluids in relative motion have been analysed by a number of investigators. The early work on the subject has been reported by Lamb (1932), and more recently by Taylor (1950), Bellman & Pennington (1954), Milne-Thomson (1952), Chandrasekhar (1961) and Rajappa & Chang (1966). The case of stagnation-point flow was never treated however. The method applied by these researchers involves determining the growth in time of a perturbation of the interface; such a technique has already been used by the author (Berghmans 1970). It shows that the stagnation-point region of the interface can be unstable and that the critical jet strength depends upon surface tension, jet size and liquid and gas densities. However, owing to the complexity of the shape of the interface the analysis has to be localized in a small region around the stagnation point. Because of this, the relation between critical jet strength and surface tension contains two constants which can not be determined by this localized technique. The present variational approach takes the whole indentation profile into account and therefore does not have this disadvantage. The variational analysis will be given in the following section, after which it will be applied to the stability problem of the impinging jet (see §3).

2. The variational approach to interface stability problems

Historically, scientists have looked at the contact surface between two different media in two ways, taking the molecular view, based on the attraction forces between molecules, or the thermodynamic view, based on ascribing an energy to the interface. The earlier molecular viewpoint culminated in the works of Laplace (1806) and Gauss (1906). However, this approach proved unsatisfactory and it was felt that some thermodynamic concepts had to be introduced to produce a more consistent theory. This was first done by Gibbs (1876, 1878), who ascribed an energy to the interface, the energy per unit area being called 'surface tension'. This tension depends only upon the two substances in contact and their thermodynamic states. It is this approach to surface phenomena which will be followed here.

Following the analysis by Tyuptsov (1966), we consider a system consisting of a closed rigid container (see figure 3) in which two stagnant immiscible fluids occupy the regions Ω_1 and Ω_2 respectively. The contact surface Σ_{12} between the two fluids (hereafter called the meniscus) intersects the container wall along the line L (the wetter perimeter). The subscripts 1 and 2 designate the fluids, while the subscript 3 refers to the container. The fluid densities are ρ_1 and ρ_2, and $\mathbf{x}$ is the position vector. By assuming that the body forces acting on the system are derivable from a potential (conservative system), one can introduce the potential energies $P_1(\mathbf{x})$ and $P_2(\mathbf{x})$ per unit volume in regions 1 and 2 respectively.

For such a system to be in stable equilibrium it is necessary that its free energy

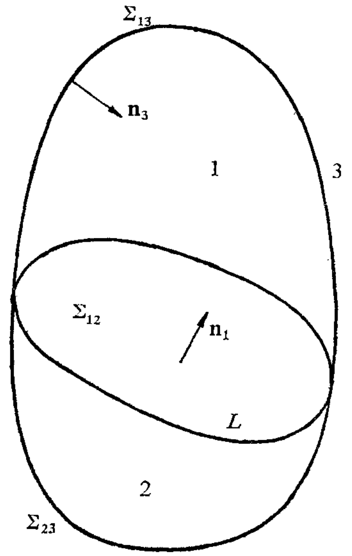

FIGURE 3. Co-ordinate system for container filled with two fluids.

be minimized (see Landau & Lifschitz 1959). The portion V of the free energy which varies because of changes at the interface is given by

$$V = \sigma_{12}\int_{\Sigma_{12}} d\Sigma + \sigma_{23}\int_{\Sigma_{23}} d\Sigma + \sigma_{13}\int_{\Sigma_{13}} d\Sigma + \int_{\Omega_1} P_1(\mathbf{x})\,d\Omega + \int_{\Omega_2} P_2(\mathbf{x})\,d\Omega, \quad (1)$$

where $\sigma_{ij} = \sigma_{ji}$ is the surface tension between media i and j, and Σ_{ij} is the contact surface between these media.

The first three terms are the contributions of the surface energy to the free energy, and the last two terms represent the contributions of the body forces. V has to be a minimum in order to have a stable equilibrium. The functional V must therefore satisfy the conditions

$$\delta V = 0, \quad \delta^2 V > 0, \quad (2), (3)$$

in which δ is the variational symbol. Condition (2) leads to Laplace's equation for capillary phenomena, from which the shape of the interface can be determined. Condition (3) ensures that the equilibrium shape determined by (2) is stable. The first and second variations of the free energy have to be expressed in terms of the displacement $\delta\mathbf{x}(\mathbf{x})$ of the fluid particles of the two media. The function $\delta\mathbf{x}(\mathbf{x})$ will be assumed sufficiently differentiable and integrable.

We now define $\mathbf{n}_3$ as the unit vector along the inward normal to the vessel, $\mathbf{n}_1$ as the unit vector perpendicular to the meniscus and directed into region 1, and $\mathbf{e}_4$ as a unit vector tangential to the wetted perimeter L and directed in a counter-clockwise direction for an observer looking from region 1. We also introduce

$$\mathbf{e}_5 = \mathbf{e}_4 \times \mathbf{n}_1, \quad \mathbf{e}_6 = \mathbf{e}_4 \times \mathbf{n}_3. \quad (4), (5)$$

These vectors are shown in figure 4 for the case of a cylindrical container.

The curvature $H(u, v)$ is defined as

$$H(u,v) = 1/2R_a(u,v), \quad (6)$$

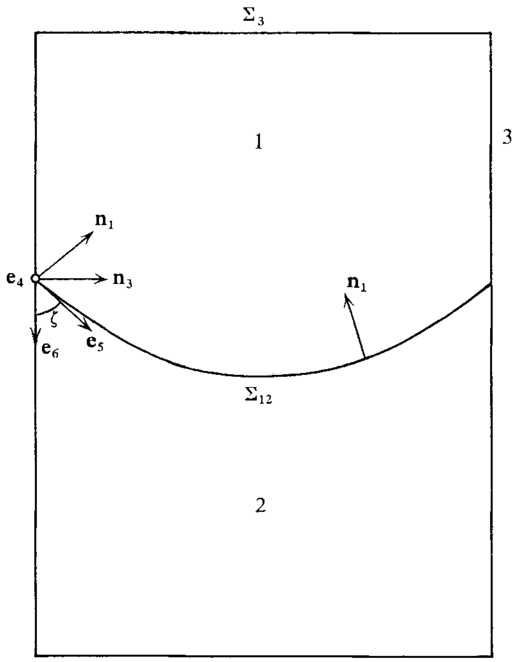

FIGURE 4. Co-ordinate system for a cylindrical container filled with two fluids.

where u and v are the curvilinear co-ordinates of the meniscus and R_a is the average radius of curvature. Tyuptsov (1966) shows that in order to satisfy (2) it is necessary that

$$p_1 - p_2 = 2\sigma_{12} H \quad \text{on} \quad \Sigma_{12} \tag{7}$$

and

$$\sigma_{12} \cos\zeta = \sigma_{31} - \sigma_{23} \quad \text{on} \quad L, \tag{8}$$

where p is the pressure and ζ the contact angle, defined by (see figure 4)

$$\cos\zeta = \mathbf{e}_5 . \mathbf{e}_6 = \mathbf{n}_1 . \mathbf{n}_3. \tag{9}$$

In order for the equilibrium shape of an interface satisfying (7) and (8) to be stable, condition (3) must be satisfied simultaneously.

The second variation of the free energy around an equilibrium shape satisfying (7) and (8) can be written as (see Tyuptsov 1966)

$$\frac{\delta^2 V}{\sigma_{12}} = \int_{\Sigma_{12}} (\tau N - \nabla^2 N) N \, d\Sigma + \int_L \left[(R_5 \cos\zeta - R_6) W + \frac{\partial N}{\partial s_5} \right] W \sin\zeta \, dl, \tag{10}$$

where

$$\tau = \frac{1}{\sigma_{12}} \frac{\partial (p_1 - p_2)}{\partial n} - 4H^2 + 2K, \tag{11}$$

$$K = 1/R_l R_k, \quad W = \delta \mathbf{x} . \mathbf{e}_6, \qquad (12), (13)$$

n = distance measured along $\mathbf{n}_1$, R_l, R_k = principal radii of curvature of the meniscus, ∇^2 = Laplace operator, s_5 = arc length along $\mathbf{e}_5$, N = displacement of Σ_{12} in the direction of $\mathbf{n}_1$, R_5 = curvature of the intersection of surface Σ_{12} with the plane containing $\mathbf{e}_5$ and $\mathbf{n}_1$, R_6 = curvature of the intersection of the container wall with the plane containing $\mathbf{e}_6$ and $\mathbf{n}_3$. Equation (10) shows the second variation to consist of two terms, the first term being due to the displacement of the meniscus and the second to the displacement of the wetted perimeter. It is the former which contains a normal derivative of the pressure difference. This accounts for the changes in potential energy of the system due to the displacement of the interface. The other parts of the meniscus term arise because of the curvature of the displacement and of the interface itself.

Since the container is rigid and both fluids are incompressible, the volume of each phase remains constant and therefore the displacement has to satisfy

$$\int_{\Sigma_{12}} \delta\mathbf{x}\,.\,\mathbf{n}_1 d\Sigma = \int_{\Sigma_{12}} N d\Sigma = 0. \tag{14}$$

The expression (10) is quadratic in the displacement and, since one is interested only in the sign of the second variation, one can normalize the displacement:

$$\int_{\Sigma_{12}} N^2 d\Sigma = 1. \tag{15}$$

The smallest value of the right-hand side of (10) is equal to the smallest value of ν in the following eigenvalue problem:

$$\tau N - \nabla^2 N - \nu N = \mu \quad \text{on} \quad \Sigma_{12}, \tag{16}$$

$$(R_5 \cos\zeta - R_6) N + (\partial N/\partial s_5) \sin\zeta = 0 \quad \text{on} \quad L. \tag{17}$$

The constant μ is chosen such that (14) is satisfied; ν is the eigenvalue of the equation. Therefore

$$(\delta^2 V)_{\min} = \sigma_{12} \nu_{\min}. \tag{18}$$

The stability problem can now be expressed in the following way: for the stability of a conservative system consisting of two immiscible fluids it is necessary that the minimum eigenvalue $\nu_{\min}$ of the problem (14)–(17) be non-negative and it is sufficient that it be positive. Attention will therefore be focused on the critical condition marking the onset of instability determined by

$$\nu_{\min} = 0. \tag{19}$$

The set of equations (14)–(17) can be simplified somewhat for an axisymmetric interface. Let r, θ and z be the usual cylindrical co-ordinates; as curvilinear co-ordinates of the meniscus we can take θ and the arc length s of the intersection of the meniscus with a plane θ = constant. If s_c and s_d are the limits of the arc length s, then the equations to be solved are

$$\int_0^{2\pi} \int_{s_c}^{s_d} N r\, ds\, d\theta = 0, \tag{20}$$

$$\int_0^{2\pi} \int_{s_c}^{s_d} N^2 r\, ds\, d\theta = 1, \tag{21}$$

$$\tau N - \frac{\partial^2 N}{\partial s^2} - \frac{1}{r}\frac{dr}{ds}\frac{\partial N}{\partial s} - \frac{\partial^2 N}{\partial \theta^2} - \nu N = \mu \quad \text{for} \quad s_c < s < s_d, \quad 0 \leqslant \theta \leqslant 2\pi \tag{22}$$

and

$$(R_5 \cos\zeta - R_6) N + \frac{\partial N}{\partial s} \sin\zeta = 0 \quad \text{at} \quad s = s_c, s_d. \tag{23}$$

If one or both ends of the curves $z(s)$ lie on the z axis, then (23) at this end-point must be replaced by the requirement of a bounded solution at this point.

It is convenient to express $N(s, \theta)$ in the following series expansion:

$$N(s,\theta) = \sum_{m=0}^{\infty} (f_m(s)\cos m\theta + g_m(s)\sin m\theta). \tag{24}$$

Substituting (24) into (22), (23) and (20) gives (with primes denoting differentiation with respect to the arc length s)

$$f_0'' + (r'/r)f_0' - \tau f_0 + \nu f_0 + \mu = 0 \quad \text{for} \quad s_c < s < s_d, \tag{25}$$

$$(R_5 \cos\zeta - R_6)f_0 + f_0' \sin\zeta = 0 \quad \text{at} \quad s = s_c, s_d, \tag{26}$$

$$\int_{s_c}^{s_d} f_0(s)r(s)\,ds = 0, \tag{27}$$

$$f_m'' + \frac{r'}{r}f_m' - \tau f_m - \frac{m^2}{r^2}f_m + \nu f_m = 0 \quad \text{for} \quad s_c < s < s_d \quad (m = 1, 2, \ldots), \tag{28}$$

$$(R_5 \cos\zeta - R_6)f_m + f_m' \sin\zeta = 0 \quad \text{at} \quad s = s_c, s_d \quad (m = 1, 2, \ldots). \tag{29}$$

Furthermore equation (21) must be satisfied.

A pair of equations identical to (28) and (29) can be written down for $g_m(s)$. Since they are identical, from now on only the functions $f_m(s)$ will be considered.

If we define $\nu_{0,1}, \nu_{0,2}, \ldots$, and in general $\nu_{m,1}, \nu_{m,2}, \ldots$, as the eigenvalues of (25)–(29) arranged in increasing order for each equation, then it is evident that

$$\nu_{\min} = \min\{\nu_{0,1}, \min(\nu_{m,1})\} \quad \text{for} \quad m = 1, 2, \ldots. \tag{30}$$

It can be shown that

$$\min(\nu_{m,1}) = \nu_{1,1} \quad \text{for} \quad m = 1, 2, \ldots, \tag{31}$$

so that

$$\nu_{\min} = \min(\nu_{0,1}, \nu_{1,1}). \tag{32}$$

The problem therefore is reduced to solving (25)–(29) with

$$m = 1 \tag{33}$$

and looking for the smallest eigenvalue of these equations. Condition (21) now becomes irrelevant since it is satisfied by all the functions f_m and g_m.

The two ordinary differential equations for $f_0(s)$ and $f_1(s)$ with the appropriate boundary conditions now have to be solved, $\tau(s)$, $r(s)$, $r'(s)$, R_5 and R_6 being determined by the shape of the interface. Therefore, first the shape of the interface satisfying (7) and (8) has to be found, after which its stability can be determined by the set of equations (25)–(29) (with $m = 1$).

The above analysis considered only stagnant fluids. However, under certain conditions this analysis is also applicable to systems with fluid flow. At the interface of fluids in relative motion, as, for instance, in the case of a gas jet impinging upon a liquid, a boundary layer is formed. In this boundary layer, close to the interface, inertia forces can be neglected. If, furthermore, viscous effects are small compared with surface tension forces, the interface can be considered to be surrounded by two layers of stagnant fluids. The pressure distribution in these layers is the same as the pressure distribution at the outer edges of the boundary layers since the pressure in a boundary layer is not dependent on distance from the wall. To study the stability of such an interface the same analysis as the one used for stagnant fluids can be used since the fluids in the immediate neighbourhood of the interface are also stagnant and only displacements of the fluids in this neighbourhood are of importance. This implies of course that instabilities caused by inertia (Kelvin–Helmholtz instabilities) or by purely viscous effects will not be detected by this approach.

In the case of the impinging gas jet, simple calculations show that viscous effects at the interface are very small compared with capillary effects under the conditions at which the instabilities have been observed. Therefore the variational approach promises to give good results for this particular problem; this will be found to be the case in the following section.

3. Stability of the interface of a gas jet impinging on a liquid

The theoretical results obtained in this section will be compared with the experimental observations of Rosler & Stewart (1968) on the stability of the interface of an air jet impinging on water. In these experiments the width of the container was about 200 times the jet radius and about 20 times the diameter of the cavity formed by the jet. For this reason the present calculations are made for an axisymmetric jet impinging on a liquid in a container whose walls are at an infinite distance from the axis of symmetry.

Using the co-ordinate system of figure 1, equation (7), which determines the shape of the indentation, can be written in the following form:

$$\frac{d^2h}{dr^2}+\frac{1}{r}\frac{dh}{dr}\left[1+\left(\frac{dh}{dr}\right)^2\right] = \frac{1}{\sigma}(p_1-p_2)\left[1+\left(\frac{dh}{dr}\right)^2\right]^{\frac{3}{2}}, \tag{34}$$

with
$$p_1-p_2 = \Delta p+(\rho_1-\rho_2)\,gh, \tag{35}$$

g = gravitational constant, Δp = over-pressure due to the impinging jet, σ = surface tension of the specific liquid–gas combination. The boundary conditions (8) become

$$dh/dr = 0 \quad \text{at} \quad r = 0, \tag{36}$$

$$h(\infty) = 0. \tag{37}$$

To determine Δp the experimental data of Gibson (1934) for laminar jets impinging on a flat plate were used. Equations (34)–(37) were solved by Rosler & Stewart (1968) by means of the Gibson data and they found good agreement between the calculated and the experimentally determined indentation profiles. Therefore Gibson's data will also be used here to determine the cavity profile. The pressure distribution Δp is approximated by

$$\Delta p = \begin{cases} p_{\max}\cos(0{\cdot}826r/r_j) & \text{for} \quad r \leqslant 1{\cdot}2r_j, \quad (38) \\ 4{\cdot}53p_{\max}\exp(-1{\cdot}76r/r_j) & \text{for} \quad r > 1{\cdot}2r_j, \quad (39) \end{cases}$$

where r_j = jet radius, $p_{\max} = \frac{1}{2}\rho_1 V_j^2$ = jet strength, V_j = maximum jet velocity. A comparison of this approximation with Gibson's data is made in figure 5 below.

As was found by Poreh & Cermak (1959), it takes a distance of 10 jet diameters downstream from the jet orifice for the mixing of jet gas with ambient gas to affect the velocity at the centre of the jet. Up to this distance the gas velocity is a function of the radial position only. Since the distance between the jet orifice and the liquid surface was less than 10 jet diameters in the experiments of Rosler & Stewart (1968), no h dependence is included in (38) and (39).

The boundary conditions (26) and (29) can be simplified in the following way. Since s_d now becomes infinite, (26) and (27) imply that f_0 and f_0' go to zero in the limit $s \to \infty$. Also since the first limit s_c is zero f_0 has to be bounded at the origin. Furthermore

$$f_0' = 0 \quad \text{at} \quad r = 0. \tag{40}$$

This condition has to be satisfied otherwise the radius of curvature of the displacement f_0 would be zero at the origin. This would give rise to infinite pressure differences and is therefore not allowed. f_1 must go to zero when $r \to \infty$ otherwise infinite forces would be required to establish this displacement, which is physically impossible. At the origin f_1 is zero because of (28).

The problem is now reduced to finding the conditions under which the two following differential equations have a solution:

$$f_0'' + (r'/r)f_0' - \tau f_0 + \mu = 0, \tag{41}$$

$$f_0'(0) = 0, \tag{42}$$

$$f_0(\infty) = f_0'(\infty) = 0, \tag{43}$$

with

$$\int_0^\infty f_0(s)\, r(s)\, dr = 0, \tag{44}$$

and

$$f_1'' + \frac{r'}{r} f_1' - \tau f_1 - \frac{1}{r^2} f_1 = 0, \tag{45}$$

$$f_1(0) = f_1'(0) = 0, \tag{46}$$

$$f_1(\infty) = f_1'(\infty) = 0. \tag{47}$$

The coefficient $\tau(s)$ appearing in (45) and (41) is a function of s since

$$\tau(s) = \frac{1}{\sigma}\frac{\partial}{\partial n}(p_1 - p_2) - \left(\frac{p_1 - p_2}{\sigma}\right)^2 + 2K, \tag{48}$$

with

$$K = \frac{d^2h}{dr^2}\frac{dh}{dr}\bigg/ r\left[1 + \left(\frac{dh}{dr}\right)^2\right]^{\frac{1}{2}}. \tag{49}$$

At each point of the interface the pressure difference $p_1 - p_2$ and its variation in a direction perpendicular to the interface can be determined by means of (34)–(39) after the shape of the indention has been determined. The radial co-ordinate and the arc length s are related by

$$\frac{ds}{dr} = \left[1 + \left(\frac{dh}{dr}\right)^2\right]^{\frac{1}{2}}. \tag{50}$$

The function $h(r)$ is of course determined by solving (34)–(37). The value of μ which assures that (27) is satisfied goes to zero when the walls of the container are at infinity (see Berghmans 1970). Therefore only solutions of the homogeneous part of (41) are required.

Owing to the complexity of the problem recourse had to be made to numerical techniques. The computer program which was written to solve this problem can be divided in two sections. The first part determines the indentation profile. An iteration procedure was required for the centre-line depth since the boundary conditions (36) and (37) are separated. This entailed choosing centre-line depths which straddled the correct value. It was found that considerable care had to be taken in choosing the finite-difference approximation to solve (34). A Runge–Kutta method as described by Collatz (1966) was used in the neighbourhood of the axis of symmetry, while for larger radial distances (larger than 150 steps) an ordinary forward-difference approximation was used. The step size did not affect the accuracy of the calculations by more than 1 % if it was smaller than 0·002 jet diameters. The step size was always kept smaller than this value.

After finding the indentation profile the differential equations (41) and (45) together with their boundary conditions were solved. Considerable simplification was obtained by taking the radial distance r as the independant variable instead of the arc length s. After substitution of (50), equations (41), (45) and (48) become

$$\frac{d^2f_0}{dr^2} = \frac{(dh/dr)\,(d^2h/dr^2)}{1+(dh/dr)^2}\frac{df_0}{dr} - \frac{1}{r}\frac{df_0}{dr} + \tau\left[1+\left(\frac{dh}{dr}\right)^2\right]f_0, \tag{51}$$

$$\frac{d^2f_1}{dr^2} = \frac{(dh/dr)\,(d^2h/dr^2)}{1+(dh/dr)^2}\frac{df_1}{dr} - \frac{1}{r}\frac{df_1}{dr} + \left[\tau+\frac{1}{r^2}\right]\left[1+\left(\frac{dh}{dr}\right)^2\right]f_1, \tag{52}$$

$$\tau = \frac{1}{\sigma}\left\{\frac{(\partial/\partial h)\,(p_1-p_2)-(\partial/\partial r)\,(p_1-p_2)\,(dh/dr)}{[1+(dh/dr)^2]^{\frac{1}{2}}}\right\} - \left(\frac{p_1-p_2}{\sigma}\right)^2 + \frac{2(dh/dr)\,(d^2h/dr^2)}{r[1+(dh/dr)^2]^2}. \tag{53}$$

Equations (51) and (52) yield the same problems for small r as equation (34). The same numerical procedure was used for all these equations.

The equations (51)–(53) can be made dimensionless by introducing two dimensionless groups: the Weber number W and Bond number B, which are defined by

$$W^2 = \rho_1 V_a^2 r_j/\sigma, \tag{54}$$

$$B^2 = (\rho_2-\rho_1)gr_j^2/\sigma, \tag{55}$$

where ρ_1 is the gas density, ρ_2 the liquid density and V_a is the average jet velocity. If for the same value of the Bond number the function f_0 diverged to large positive values for one Weber number and to large negative values for another Weber number, it was decided that the critical Weber number had to lie somewhere between these two Weber numbers. The same is true for f_1. By narrowing these limits it was possible to determine the critical Weber number, belonging to a certain Bond number, to within 5 % of its nominal value. It was the function f_0 which determined the onset of instability. The Weber numbers which gave convergent solutions of f_1 were always larger than the ones which gave convergent solutions of f_0 (for the same Bond number). Further details about the computer program can be found in Berghmans (1970).

Figure 5 gives a typical indention profile with the corresponding $f_0(r)$ distribution. The results of the numerical analysis are plotted in figure 6 in terms of

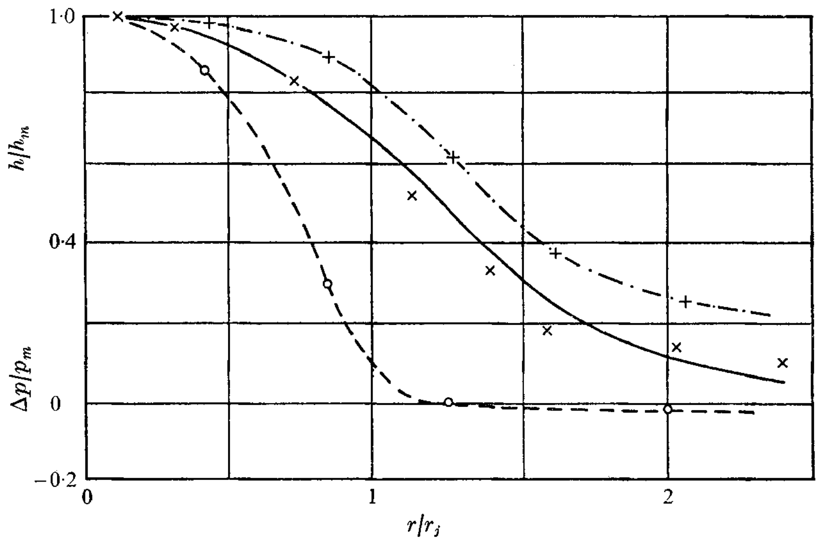

FIGURE 5. Pressure distribution of a circular free jet impinging on a flat plate. ×, data of Gibson (1934); —, equations (38) and (39). Analytical results for $B = 0{\cdot}47$, $W = 1{\cdot}3$, $h_m = 5{\cdot}3\,r_j$: +, normalized indentation depth; ○, $f_0(r)$.

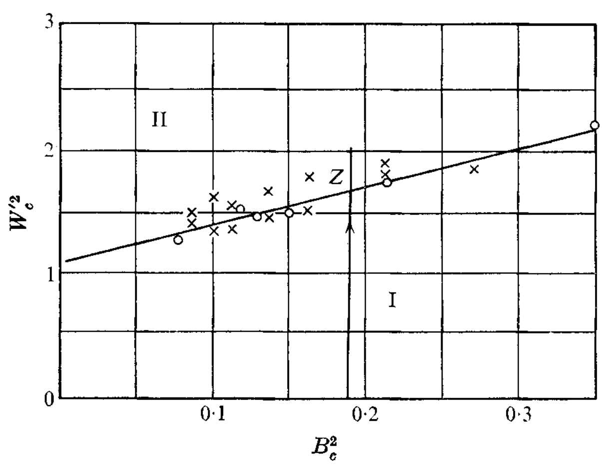

FIGURE 6. Critical Weber number versus critical Bond number for a gas jet impinging on a liquid. ×, data of Rosler & Stewart (1968); ○, analytical results; —, equation (56).

critical Bond number and critical Weber number. As can be seen from this figure the present results fall within the range of accuracy of the experimental data of Rosler & Stewart (1968), which is rather surprising. The stability calculations are very sensitive to the assumed pressure distribution in the neighbourhood of the stagnation point. The choice of the approximation to the actual pressure distribution taken in the present calculations is therefore very good. A good approximation to both the experimental and the analytical results is given by

$$W_c^2 = 1{\cdot}04 + 3{\cdot}3B_c^2, \tag{56}$$

which is shown in figure 6.

For a certain liquid–gas combination, jet diameter and jet separation, increasing the jet velocity will cause an increase in W so that W follows the vertical line in figure 6. The interface will remain stable as long as one remains in region I. On reaching point Z (see figure 6) the interface becomes unstable and will remain so as long as one stays in region II.

Equation (56) can be written as

$$V_{a,c}^2 = (1/\rho_1)\,[3{\cdot}3(\rho_2-\rho_1)gr_j + 1{\cdot}04\sigma/r_j]. \tag{57}$$

This equation shows that increasing the liquid density ρ_2 or the surface tension σ is stabilizing, but increasing the gas density ρ_1 is destabilizing. The effect of the jet radius is more complex. $V_{a,c}^2$ reaches a minimum at $r_{j,\,\mathrm{min}}$ determined by

$$r_{j,\,\mathrm{min}} = 0{\cdot}56[\sigma/(\rho_2-\rho_1)g]^{\frac{1}{2}}. \tag{58}$$

Therefore increasing the jet radius for values smaller than $r_{j,\,\mathrm{min}}$ is destabilizing, while for values larger than $r_{j,\,\mathrm{min}}$ it is stabilizing.

4. Conclusions

The results of the present work can be summarized as follows.

(i) The variational approach to stability problems, which had already been applied to stagnant fluids, was found to be also valid for systems in which the fluids are in motion. However one has to limit oneself to problems in which inertia and viscous effects are small compared with surface tension effects.

(ii) Solution of the stability problem of a jet–liquid interface by the variational approach leads to results which fall within the range of accuracy of the experimental data. It was found that increasing surface tension or liquid density has a stabilizing effect, while increasing the gas density or jet velocity is destabilizing. A jet radius $r_{j,\,\mathrm{min}}$ for which the critical jet velocity reaches a minimum exists and, for radii smaller than $r_{j,\,\mathrm{min}}$, increasing the jet radius is destabilizing, while it is stabilizing for radii larger than $r_{j,\,\mathrm{min}}$.

In the present analysis a pressure distribution from which the interface could be determined was assumed; a more complete analysis would have to solve the flow in both fluids simultaneously. This leads to serious problems since the interface is a deformable surface. The results of the application of the variational method to the jet problem show the method to be very successful. Remarkable results were also obtained for the stability problem of small gas bubbles moving through liquids because of their own buoyancy (see Berghmans 1970). This is not surprising since, because of its generality, this method has a wide range of applicability.

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